

# On Finite $J$ -Hermitian Quantum Mechanics

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## Abstract

In his recent paper [4], the author discussed  $J$ -Hermitian quantum mechanics and showed that  $PT$ -symmetric quantum mechanics is essentially  $J$ -Hermitian quantum mechanics. In this paper, the author discusses finite  $J$ -Hermitian quantum mechanics which is derived naturally from its continuum one and its relationship with finite  $PT$ -symmetric quantum mechanics.

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## INTRODUCTION. $J$ -HERMITIAN QUANTUM MECHANICS

Let  $\mathcal{K}$  be the complex vector space spanned by the eigenstates of a quantum system determined by Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi. \quad (1)$$

Due to the boundary conditions on state functions  $\psi(x, t)$ , without loss of generality we may assume that  $\mathcal{K}$  is separable. Let  $J : \mathcal{K} \rightarrow \mathcal{K}$  be an involution i.e.  $J^2 = I$ , the identity map. Define an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{K}$  as follows: For any  $\varphi, \psi \in \mathcal{K}$ ,

$$\begin{aligned} \langle \varphi, \psi \rangle &:= \langle \varphi | J | \psi \rangle \\ &= \int_{-\infty}^{\infty} \bar{\varphi} J \psi dx, \end{aligned} \quad (2)$$

where  $\langle \cdot | \cdot \rangle$  denotes Dirac bracket. We are interested in a particular  $J = P$ , the parity. That is  $J$  acts on  $\psi(x, t)$  as

$$J\psi(x, t) = \psi(-x, t). \quad (3)$$

Then the inner product is written

$$\langle \varphi, \psi \rangle = \int_{-\infty}^{\infty} \bar{\varphi}(x) \psi(-x) dx. \quad (4)$$

The inner product is a Hermitian product, i.e. it satisfies

IP1.  $\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle}$  for any  $\varphi, \psi \in \mathcal{K}$ .

IP2.  $\langle \varphi, a\psi_1 + b\psi_2 \rangle = a\langle \varphi, \psi_1 \rangle + b\langle \varphi, \psi_2 \rangle$  for any  $\varphi, \psi_1, \psi_2 \in \mathcal{K}$  and  $a, b \in \mathbb{C}$ .

The squared norm  $\|\psi\|^2 = \int_{-\infty}^{\infty} \bar{\psi}(x) \psi(-x) dx$  can be positive, zero, or negative, so  $\langle \cdot, \cdot \rangle$  is an indefinite Hermitian product. The space  $\mathcal{K}$  may be decomposed to

$$\mathcal{K} = \mathcal{K}^+ \dot{+} \mathcal{K}^-,$$

where  $\mathcal{K}^+$  is spanned by the eigenstates  $\psi_n^+$ ,  $n = 1, 2, \dots$  such that  $\langle \psi_m^+, \psi_n^+ \rangle = \delta_{mn}$  and  $\mathcal{K}^-$  is spanned by the eigenstates  $\psi_n^-$ ,  $n = 1, 2, \dots$  such that  $\langle \psi_m^-, \psi_n^- \rangle = -\delta_{mn}$ . Here,  $\dot{+}$  stands for orthogonal direct sum.  $\mathcal{K}$  cannot be a (pre-)Hilbert space. It is called a Krein space in mathematics literature [3].  $J$  is required to satisfy: for each  $n = 1, 2, \dots$ ,

$$\begin{aligned} J\psi_n^+(x, t) &= \psi_n^+(-x, t) = \psi_n^+(x, t); \\ J\psi_n^-(x, t) &= \psi_n^-(-x, t) = -\psi_n^-(x, t). \end{aligned} \quad (5)$$

The equations (5) indicate that  $\psi_n^+(x, t)$  are even functions with respect to  $x$ , while  $\psi_n^-(x, t)$  are odd functions with respect to  $x$ . An involution on a Krein space satisfying (5) is called the *fundamental symmetry* in mathematics literature [3]. The fundamental symmetry  $J$  is used to define a positive definite inner product  $\langle \cdot, \cdot \rangle_J$  on the Krein space  $\mathcal{K}$ : for any  $\varphi, \psi \in \mathcal{K}$ ,

$$\langle \varphi, \psi \rangle_J := \langle \varphi, J\psi \rangle = \langle \varphi | \psi \rangle. \quad (6)$$

This inner product (6) is called the  $J$ -inner product [3]. This  $J$ -inner product allows us to avoid awkward negative probability that was resulted by the indefinite Hermitian product (4).  $\mathcal{K}$  together with the  $J$ -inner product becomes a separable pre-Hilbert space and hence we may consider the Hilbert space of states as its completion.

Let  $A : \mathcal{K} \longrightarrow \mathcal{K}$  be a bounded linear operator on a Krein space  $\mathcal{K}$ . Then we may define its *adjoint*  $A^*$  analogously to that in the standard Hermitian quantum mechanics. That is,  $A^* : \mathcal{K} \longrightarrow \mathcal{K}$  is a linear operator that satisfies the property

$$\langle \varphi, A\psi \rangle = \langle A^*\varphi, \psi \rangle \quad (7)$$

for all  $\varphi, \psi \in \mathcal{K}$ . The adjoint satisfies the properties:

$$\text{A1. } (A + B)^* = A^* + B^*,$$

$$\text{A2. } (\lambda A)^* = \bar{\lambda} A^*,$$

$$\text{A3. } (AB)^* = B^* A^*,$$

$$\text{A4. } A^{**} = A,$$

$$\text{A5. if } A \text{ is invertible, then } (A^{-1})^* = (A^*)^{-1}.$$

If we denote by  $A^{[*]}$  the adjoint of  $A$  with respect to the  $J$ -inner product (6), then  $A^*$  and  $A^{[*]}$  are related by ([3])

$$JA^{[*]}J = A^*. \quad (8)$$

A bounded linear operator  $A : \mathcal{K} \longrightarrow \mathcal{K}$  is said to be *self-adjoint* or  *$J$ -Hermitian* or simply *Hermitian* (in case there is no confusion with the notion of ordinary Hermitian operators) if  $A = A^*$ , i.e. for any  $\varphi, \psi \in \mathcal{K}$

$$\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle \quad (9)$$

or

$$\int_{-\infty}^{\infty} (\overline{A\varphi}) J\psi dx = \int_{-\infty}^{\infty} \bar{\varphi} J(A\psi) dx. \quad (10)$$

It turns out that a Hamiltonian of the form

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x), \quad (11)$$

where the potential  $V(x)$  acts on  $\psi(x, t)$  by multiplication, is  $J$ -Hermitian if and only if it is  $PT$ -symmetric [4]. Since  $J$ -Hermitian operators are guaranteed to have all real eigenvalues, so are  $PT$ -symmetric Hamiltonians. In [4], the author also shows that the time evolution of a state function determined by the Hamiltonian  $\hat{H}$  in (11) is unitary if and only if  $\hat{H}$  is  $PT$ -symmetric.

In the following sections, the author will deduce the notion of  $J$ -Hermitian matrices as Hamiltonians and discuss the relationship between  $J$ -Hermitian Hamiltonians and  $PT$ -symmetric Hamiltonians, and their physical implications.

## **$J$ -HERMITIAN MATRICES**

Let  $\mathbb{C}^2$  denote the complex vector space

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

For any  $v, w \in \mathbb{C}^2$ , define an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{C}^2$  by

$$\begin{aligned} \langle v, w \rangle &:= \langle v | J | w \rangle \\ &= v^\dagger J w, \end{aligned} \quad (12)$$

where  $v^\dagger = \bar{v}^t$ . Here  $\langle v | w \rangle$  stands for Dirac bracket which is the standard positive definite Hermitian product  $v^\dagger w$  and  $J : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is an involution i.e.  $J^2 = I$ , the identity transformation. Either  $J = I$  or, up to diagonalisation and sign,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (13)$$

since  $J$  has eigenvalues  $\pm 1$ . If  $J = I$ , then (12) is simply Dirac bracket. In this paper, we consider  $J$  in (13). Then  $\langle \cdot, \cdot \rangle$  is an indefinite Hermitian product on  $\mathbb{C}^2$ . Carl M. Bender (for example [1], [2]) introduced the inner product (12) as  $(PTv)^t w$  where  $P$  and  $T$  stand for parity and time-reversal operator, respectively. Parity  $P$  is defined to be  $J$  in (13). Carl

M. Bender defined  $P$  by  $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Since  $P$  has eigenvalues  $\pm 1$ ,  $J$  is the diagonalisation of the parity used by Bender. Time-reversal operator  $T$  is defined as complex conjugation. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$\langle e_1, e_1 \rangle = 1, \quad \langle e_1, e_2 \rangle = 0, \quad \langle e_2, e_2 \rangle = -1.$$

Thus,  $\{e_1, e_2\}$  is an orthonormal basis of  $\mathbb{C}^2$  with respect to the indefinite Hermitian product (12). The involution  $J$  (13) satisfies

$$\begin{aligned} J e_1 &= e_1, \\ J e_2 &= -e_2. \end{aligned} \tag{14}$$

So,  $J$  is the fundamental symmetry and  $(\mathbb{C}^2, J)$  is a 2-dimensional Krein space. Let us define another inner product  $\langle \cdot, \cdot \rangle_J$  as follows. For any  $v, w \in \mathbb{C}^2$ ,

$$\langle v, w \rangle_J := \langle v, Jw \rangle. \tag{15}$$

Then  $\langle v, w \rangle_J = \langle v | w \rangle = v^\dagger w$  i.e. the usual Dirac bracket. We call  $\langle \cdot, \cdot \rangle_J$  *J-inner product*.  $\mathbb{C}^2$  with  $J$ -inner product is a 2-dimensional Hilbert space.

Let  $\hat{H} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be a linear operator on  $\mathbb{C}^2$ . We find the matrix representation of the adjoint  $\hat{H}^*$ . First write the matrix representation of  $\hat{H}$  as  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  where  $a, b, c, d \in \mathbb{C}$ . Then

$$\hat{H} e_1 = a e_1 + b e_2,$$

$$\hat{H} e_2 = c e_1 + d e_2.$$

So, we obtain

$$a = \langle e_1, \hat{H} e_1 \rangle, \quad b = -\langle e_2, \hat{H} e_1 \rangle,$$

$$c = \langle e_1, \hat{H} e_2 \rangle, \quad d = -\langle e_2, \hat{H} e_2 \rangle.$$

Let  $\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$  be the matrix representation of the adjoint  $\hat{H}^*$ . Then

$$\hat{H}^* e_1 = a^* e_1 + b^* e_2,$$

$$\hat{H}^* e_2 = c^* e_1 + d^* e_2.$$

So, we obtain

$$\begin{aligned} a^* &= \langle e_1, \hat{H}^* e_1 \rangle, \quad b^* = -\langle e_2, \hat{H}^* e_1 \rangle, \\ c^* &= \langle e_1, \hat{H}^* e_2 \rangle, \quad d^* = -\langle e_2, \hat{H}^* e_2 \rangle. \end{aligned}$$

Now,

$$\begin{aligned} a^* &= \langle e_1, \hat{H}^* e_1 \rangle = \langle \hat{H} e_1, e_1 \rangle = \overline{\langle e_1, \hat{H} e_1 \rangle} = \bar{a}, \\ b^* &= -\langle e_2, \hat{H}^* e_1 \rangle = -\langle \hat{H} e_2, e_1 \rangle = -\overline{\langle e_1, \hat{H} e_2 \rangle} = -\bar{c}, \\ c^* &= \langle e_1, \hat{H}^* e_2 \rangle = \langle \hat{H} e_1, e_2 \rangle = \overline{\langle e_2, \hat{H} e_1 \rangle} = -\bar{b}, \\ d &= -\langle e_2, \hat{H} e_2 \rangle = -\langle \hat{H} e_2, e_2 \rangle = -\overline{\langle e_2, \hat{H} e_2 \rangle} = \bar{d}. \end{aligned}$$

Thus, the matrix representation of the adjoint  $\hat{H}^*$  is given by

$$\hat{H}^* = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}. \quad (16)$$

If  $\hat{H}$  is  $J$ -Hermitian i.e.  $\hat{H} = \hat{H}^*$ , then  $\hat{H}$  is

$$\hat{H} = \begin{pmatrix} a & b \\ -\bar{b} & d \end{pmatrix}, \quad (17)$$

where  $a$  and  $d$  are real. We call a matrix of the form (17) a  $J$ -Hermitian matrix.  $2 \times 2$   $J$ -Hermitian matrices can be characterised as follows.

**Theorem 1** *A  $2 \times 2$  matrix  $\hat{H}$  is  $J$ -Hermitian if and only if*

$$J\hat{H}^\dagger J = \hat{H}. \quad (18)$$

(18) is indeed identical to (8). In general, we have

**Theorem 2** *An  $n \times n$  matrix  $\hat{H}$  is  $J$ -Hermitian if and only if*

$$J\hat{H}^\dagger J = \hat{H}, \quad (19)$$

where

$$J = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

**Example 1** If a linear operator  $\hat{H} : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$  is  $J$ -Hermitian, then  $\hat{H}$  may be written as

$$\hat{H} = \begin{pmatrix} a & b & c \\ \bar{b} & d & e \\ -\bar{c} & -\bar{e} & f \end{pmatrix} \quad (20)$$

where  $a$ ,  $d$ , and  $f$  are real.

## TIME EVOLUTION

Let  $\hat{H}$  be a  $2 \times 2$  complex matrix. The Schrödinger equation

$$i\hbar \frac{d\psi(t)}{dt} = \hat{H}\psi(t) \quad (21)$$

has solution

$$\psi(t) = \hat{U}(t)\psi(0) \quad (22)$$

where

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right). \quad (23)$$

Suppose that  $\langle\psi(0), \psi(0)\rangle \neq 0$ . Physically we want  $\hat{U}(t)$  to be unitary, i.e.

$$\begin{aligned} \langle\psi(t), \psi(t)\rangle &= \langle\hat{U}(t)\psi(0), \hat{U}(t)\psi(0)\rangle \\ &= (\hat{U}(t)\psi(0))^\dagger J \hat{U}(t)\psi(0) \\ &= \psi^\dagger(0) \hat{U}^\dagger(t) J \hat{U}(t) \psi(0) \\ &= \psi^\dagger(0) J \psi(0) \\ &= \langle\psi(0), \psi(0)\rangle. \end{aligned}$$

Since  $\langle\psi(0), \psi(0)\rangle \neq 0$ , we see that  $\hat{U}(t)$  is unitary if and only if

$$\hat{U}^\dagger(t) J \hat{U}(t) = J. \quad (24)$$

The set of unitary transformations i.e. transformations of  $\mathbb{C}^2$  that satisfy (24) forms a Lie subgroup of  $SL(2, \mathbb{C})$  which is the well-known indefinite unitary group  $U(1, 1)$ . This is consistent with the fact that the structure group of the frame bundle  $LC^2$  is  $U(1, 1)$  when  $\mathbb{C}^2$  is considered as an indefinite 2-dimensional Hermitian manifold. Since  $J = J^{-1}$ ,

$$J \hat{U}^\dagger(t) J \hat{U}(t) = \exp\left(\frac{i}{\hbar} J \hat{H}^\dagger J t\right) \exp\left(-\frac{i}{\hbar} \hat{H} t\right). \quad (25)$$

Clearly, if  $J\hat{H}^\dagger J = \hat{H}$  then  $\hat{U}(t)$  is unitary. Conversely, if  $\hat{U}(t)$  is unitary then by differentiating (25) at  $t = 0$ , we obtain  $J\hat{H}^\dagger J = \hat{H}$ . Thus, we have

**Theorem 3**  $\hat{U}(t)$  in (23) is unitary if and only if  $\hat{H}$  is  $J$ -Hermitian.

**Remark 1** A  $2 \times 2$  matrix  $\hat{H}$  is  $J$ -Hermitian if and only if  $-i\hat{H} \in \mathfrak{u}(1,1)$ , the Lie algebra of  $U(1,1)$ . Any indefinite Hermitian manifold is orientable so the structure group of the frame bundle  $LC^2$  may be reduced to the special indefinite unitary group  $SU(1,1)$ . The Lie algebra  $\mathfrak{su}(1,1)$  is the set of elements in  $\mathfrak{u}(1,1)$  that are trace-free. With the additional condition  $\text{tr}(\hat{H}) = 0$ , (17) may be written as

$$\hat{H} = \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix} \quad (26)$$

where  $a$  is real.

## THE REALITY OF EIGENVALUES

Let  $\hat{H}$  be a  $J$ -Hermitian matrix and  $E$  an eigenvalue of  $\hat{H}$ . Suppose that  $E$  is a non-real complex number  $E = \alpha + i\beta$  where  $\alpha$  and  $\beta \neq 0$  are real. Let  $v$  be an eigenvector of  $\hat{H}$  with the eigenvalue  $E$ . Then  $\hat{H}v = Ev$ . Multiply this equation by  $\bar{E}$ . Then we get  $\bar{E}\hat{H}v = |E|^2v$  or  $\frac{\bar{E}}{|E|^2}\hat{H}v = v$ . Now,

$$\begin{aligned} ||v||^2 &= v^\dagger Jv \\ &= \left[ \frac{\bar{E}}{|E|^2} \hat{H}v \right]^\dagger Jv \\ &= \frac{E}{|E|^2} v^\dagger \hat{H}^\dagger Jv \\ &= \frac{E}{|E|^2} v^\dagger J\hat{H}v \\ &= \frac{E^2}{|E|^2} ||v||^2. \end{aligned}$$

From this we obtain  $(E^2 - |E|^2)||v||^2 = 0$ . Since  $E$  is a non-real complex number,  $E^2 \neq |E|^2$  and hence  $||v||^2 = 0$ . Note that the inner product is indefinite, so  $||v||^2 = 0$  does not mean that  $v = O$ . Hence, the eigenvalues of a  $J$ -Hermitian matrix are all real as long as the eigenvectors they are belonging to have non-vanishing squared norms. Furthermore, an



eigenvector with zero eigenvalue must have vanishing norm. To see this, let  $v = (\alpha, \beta)^t$  be an eigenvector with zero eigenvalue. Then the equation  $\hat{H}v = 0$  may be written as

$$a\alpha + b\beta = 0, \quad (27)$$

$$-\bar{b}\alpha - a\beta = 0. \quad (28)$$

If (27) and (28) are not equivalent or  $\det \hat{H} \neq 0$ , then  $v = (0, 0)^t$ . If (27) and (28) are equivalent or  $\det \hat{H} = 0$ , then  $v$  may be chosen to be  $v = (-b, a)^t$ . Now,

$$\begin{aligned} ||v||^2 &= \begin{pmatrix} -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} \\ &= |b|^2 - a^2 \\ &= \det \hat{H} \\ &= 0. \end{aligned}$$

The characteristic equation  $\det(\hat{H} - EI) = 0$  is equivalent to  $E^2 = a^2 - |b|^2$  where  $E$  is an eigenvalue of  $\hat{H}$ . We require that each eigenvalue  $E$  is non-zero real, so the  $J$ -Hermitian Hamiltonian  $\hat{H}$  must satisfy

$$\det \hat{H} < 0. \quad (29)$$

$\hat{H}$  has two distinct real eigenvalues, one positive and the other negative

$$E_{\pm} = \pm \sqrt{a^2 - |b|^2}. \quad (30)$$

Let  $v_{\pm}$  denote eigenvectors with eigenvalues  $E_{\pm}$ , respectively. The equation  $\hat{H}v = Ev$  is written as

$$(a - E)\alpha + b\beta = 0 \quad (31)$$

$$-\bar{b}\alpha - (a + E)\beta = 0 \quad (32)$$

where  $v = (\alpha, \beta)^t$ . Without loss of generality we may assume that  $a > 0$ . Denote by  $v_+$  and  $v_-$  eigenvectors with eigenvalues  $E_+$  and  $E_-$ , respectively. From the equation (31),  $v_+$  may be chosen to be  $v_+ = (-b, a - E_+)^t$ . Then

$$\begin{aligned} ||v_+||^2 &= \begin{pmatrix} -\bar{b} & a - E_+ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -b \\ a - E_+ \end{pmatrix} \\ &= |b|^2 - (a - E_+)^2 \\ &= -2E_+(E_+ - a). \end{aligned}$$

From the equation (32),  $v_-$  may be chosen to be  $v_- = (a + E_-, -\bar{b})^t$ . Then

$$\begin{aligned} ||v_-||^2 &= \begin{pmatrix} a + E_- & -b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a + E_- \\ -\bar{b} \end{pmatrix} \\ &= (a + E_-)^2 - |b|^2 \\ &= 2E_-(E_- + a). \end{aligned}$$

From (30) we find that  $E_+ - a < 0$  and  $E_- + a > 0$ . Thus  $v_+$  has a positive squared norm and  $v_-$  has a negative squared norm. Moreover,  $v_+$  and  $v_-$  are orthogonal to each other:

$$\begin{aligned} \langle v_+, v_- \rangle &= \begin{pmatrix} -\bar{b} & a - E_+ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a + E_- \\ -\bar{b} \end{pmatrix} \\ &= \begin{pmatrix} -\bar{b} & -(a - E_+) \end{pmatrix} \begin{pmatrix} a + E_- \\ -\bar{b} \end{pmatrix} \\ &= -\bar{b}(a + E_-) + \bar{b}(a - E_+) \\ &= 0. \end{aligned}$$

The zero value in the last line is obtained by (30).

## ***J*-HERMITIAN AND *PT*-SYMMETRIC HAMILTONIANS**

In this section, we study the relationship between *J*-Hermitian and *PT*-symmetric Hamiltonians.

Let  $\hat{H}$  be a  $2 \times 2$  complex matrix. The parity  $P = J$  acts on  $\hat{H}$  as  $P\hat{H}P^{-1}$ . As mentioned earlier, time-reversal operator  $T$  is defined to be complex conjugation.  $\hat{H}$  is said to be *PT*-symmetric if

$$P\overline{\hat{H}}P^{-1} = \hat{H}. \quad (33)$$

Suppose that  $\hat{H}$  is *PT*-symmetric. In addition, we must require that  $\hat{H}$  is symmetric. Otherwise, time evolution determined by the *PT*-symmetric Hamiltonian would not be unitary. Then  $\hat{H}$  is of the form

$$\hat{H} = \begin{pmatrix} a & ib \\ ib & c \end{pmatrix},$$

where  $a, b, c$  are real numbers. Thus,  $\hat{H}$  is also *J*-Hermitian. *PT*-symmetric Hamiltonians are a special class of *J*-Hermitian Hamiltonians.

## TIME EVOLUTION WITH $J$ -INNER PRODUCT

In  $J$ -Hermitian quantum mechanics, negative probabilities are merely an artifact of the indefinite Hermitian inner product (12), and there appears to be no physically meaningful notion for negative probabilities. So one may argue that we must use  $J$ -inner product (15) for doing physics with  $J$ -Hermitian quantum mechanics. Since  $J$ -inner product is the same as the usual Dirac bracket, time evolution operator (23) is unitary with respect to  $J$ -inner product if and only if Hamiltonian is Hermitian in ordinary sense i.e.  $\hat{H}^\dagger = \hat{H}$ . Thus, in order for time evolution determined by a  $J$ -Hermitian Hamiltonian  $\hat{H}$  to be unitary with respect to  $J$ -inner product, it is also required to be Hermitian in ordinary sense which results  $\hat{H}$  in the form

$$\hat{H} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad (34)$$

i.e. a real diagonal matrix. Clearly  $\pm a$  are the eigenvalues of  $\hat{H}$ .

On the other hand, since a  $J$ -Hermitian matrix  $\hat{H}$  has 2 distinct real eigenvalues, it is always diagonalisable i.e. there exists a  $2 \times 2$  invertible matrix  $\hat{M}$  such that

$$\hat{M}^{-1} \hat{H} \hat{M} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where  $\lambda_1, \lambda_2$  are the two distinct real eigenvalues of  $\hat{H}$ . In fact, the matrix  $\hat{M}$  is given by the block matrix of its column vectors  $v_1, v_2$

$$\hat{M} = \begin{pmatrix} v_1 & v_2 \end{pmatrix},$$

where  $v_1, v_2$  are eigenvectors with the eigenvalues  $\lambda_1, \lambda_2$ , respectively. If  $v_1, v_2$  are orthonormal vectors such that  $\langle v_1, v_1 \rangle = 1$ ,  $\langle v_1, v_2 \rangle = 0$ , and  $\langle v_2, v_2 \rangle = -1$ , then  $\hat{M}^\dagger J \hat{M} = J$  i.e.  $\hat{M}$  is unitary. If  $\hat{H}$  is  $J$ -Hermitian, then for any unitary matrix  $\hat{M}$ ,  $\hat{U}(t) = \exp\left(-\frac{i}{\hbar} \hat{M}^{-1} \hat{H} \hat{M} t\right)$  is unitary and so  $\hat{M}^{-1} \hat{H} \hat{M}$  is  $J$ -Hermitian.

Both Hamiltonians  $\hat{H}$  and its diagonalisation  $\hat{M}^{-1} \hat{H} \hat{M}$  have the same eigenvalues (observables) and physically the quantum mechanical systems from the two Hamiltonians are not distinguishable. Therefore, as long as time-independent Hamiltonians are concerned, finite  $J$ -Hermitian or equivalently  $PT$ -symmetric quantum mechanics is essentially the standard Hermitian quantum mechanics.

## THE SYMMETRY OF $J$ -HERMITIAN QUANTUM MECHANICS

As seen earlier, the set of unitary transformations of the 2-dimensional Krein space  $\mathbb{C}^2$  forms indefinite unitary group  $U(1,1)$ . Hence,  $U(1,1)$  is the symmetry group of the 2-dimensional Krein space  $\mathbb{C}^2$ . As discussed in the preceding section, time evolution is also required to be unitary with respect to  $J$ -inner product which is the usual Dirac bracket. The symmetry group of  $\mathbb{C}^2$  as a 2-dimensional Hilbert space is  $U(2)$ . Thus,  $J$ -Hermitian quantum mechanics must have  $U(1,1) \cap U(2)$  symmetry. Suppose that  $U \in U(1,1) \cap U(2)$ . Then

$$U^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (35)$$

and

$$U^\dagger U = I. \quad (36)$$

Multiplying (35) by  $U$  from the left, we obtain

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (37)$$

Let  $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  where  $a, b, c, d \in \mathbb{C}$ . Then by (37), we find that  $b = c = 0$  and so

$$U = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}. \quad (38)$$

Since  $U^\dagger U = I$ ,  $a$  and  $d$  satisfy

$$|a|^2 = |d|^2 = 1. \quad (39)$$

Conversely, any complex matrix  $U$  of the form (38) satisfying (39) is in the intersection  $U(1,1) \cap U(2)$ . Hence,

$$\begin{aligned} U(1,1) \cap U(2) &= \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix} : 0 \leq \theta, \phi \leq 2\pi \right\} \\ &\cong \{(e^{i\theta}, e^{i\phi}) : 0 \leq \theta, \phi \leq 2\pi\} \\ &= SO(2) \times SO(2). \end{aligned}$$

That is,  $J$ -Hermitian quantum mechanics has toroidal symmetry. When  $\mathbb{C}^2$  is considered as a 2-dimensional Hermitian or indefinite Hermitian manifold, the symmetry group of  $\mathbb{C}^2$  as a 2-dimensional Hilbert or Krein space is the same as the structure group of the frame bundle  $L\mathbb{C}^2$ . Since a Hermitian manifold is orientable, the structure group may be reduced to  $SU(2)$  or  $SU(1,1)$  depending on the signature of  $\mathbb{C}^2$ . As a result, the symmetry of  $J$ -Hermitian quantum mechanics may be reduced to  $SU(1,1) \cap SU(2) = SO(2)$ . Although finite  $J$ -Hermitian or  $PT$ -symmetric quantum is essentially the standard Hermitian quantum mechanics, geometrically it exhibits a distinct symmetry.

## CONCLUSION

The author derived the notion of  $J$ -Hermitian matrices naturally from the notion of  $J$ -Hermitian Hamiltonians he discussed in [4].  $J$ -Hermitian matrices may serve as Hamiltonians for finite dimensional quantum mechanical systems. When  $\mathbb{C}^2$  is equipped with standard indefinite Hermitian product (12) with  $J$  in (13), time evolution of a state vector determined by a Hamiltonian is unitary if and only if the Hamiltonian is  $J$ -Hermitian. It turns out that a  $PT$ -symmetric Hamiltonian  $\hat{H}$  is  $J$ -Hermitian if  $\hat{H}$  is also symmetric. Note that in continuum case [4], a  $PT$ -symmetric Hamiltonian is automatically symmetric because it is self-adjoint. The author argued that finite  $J$ -Hermitian or  $PT$ -symmetric quantum mechanics is essentially the standard Hermitian quantum mechanics as long as time-independent Hamiltonians are concerned. Hence there is no issue with unitarity of time evolution with  $J$ -inner product unlike its continuum case [4].

The author's colleague and a physicist Lawrence R. Mead observed the violation of unitarity of a time evolution with  $J$ -inner product when concerned  $J$ -Hermitian or  $PT$ -symmetric Hamiltonian is time-dependent [5]. This is perhaps due to the fact that time-dependent  $J$ -Hermitian Hamiltonian  $\hat{H}(t)$  is not diagonalisable except when  $t = 0$ . Therefore, the only physically acceptable time-dependent  $J$ -Hermitian Hamiltonians appear to be the diagonal ones. This issue will be investigated further.

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